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ON THE CONVERGENCE OF THE FINITE ELEMENT METHOD FOR PROBLEMS WITH SINGULARITY

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Abstract—For problems with singularities, the convergence rate for the finite element method are often controlled by the nature of the solution near the points of singularity. Unless the singularities are properly handled, the regular so-called high-accuracy element will not be able to improve the rate of convergence.

INTRODUCTION

IT IS well known that in the finite element formulation by assumed displacements when the compatibility conditions are satisfied and the constant strain approximations can be accomplished as the size of the elements tends to zero, the convergence of the finite element solution can be proved [1–6] and the rate of convergence can be established. In determining the rate of convergence, it has generally been implicitly or explicitly assumed that the exact solution is analytic or at least sufficiently smooth so that it can be approximated by a polynomial. If the interpolation function of the finite element formulation is a complete polynomial of degree p, the error in the approximation of a smooth function is of order h^{p+1} where h is the size of the elements. Such convergence criterion is, of course, not applicable when the exact solution contains singularities. In Ref. [7] it has shown that the convergence rate in energy is of order $h^{2\alpha}$ for a plane elasticity problem with a stress singularity $r^{\alpha-1}$ when rectangular elements with bilinear interpolation functions or constant strain triangular elements are used. The purpose of the present paper is to show that for an elasticity problem, in general, the rate of convergence of the finite element solution in the presence of singularities often controlled by the nature of the singularities. This paper also establishes the convergence rate of the stress intensity factor in the plane crack problems.

THE RATE OF CONVERGENCE

To establish the convergence proof, we shall assume that $\mathbf{u}_0$ is the exact solution of the problem over a domain A. For simplicity, we shall assume that the solution is sufficiently smooth except that

$$\mathbf{u}_0 = r^\alpha \mathbf{g}(\mathbf{x}) \tag{1}$$

near the point of singularity, say point R in A. In equation (1), r is the radial distance from R, $\mathbf{g}$ is a smooth function, $\mathbf{x}$ is the spatial coordinates, α is not an integer and

$$p+1 > \alpha+\frac{n}{2} > 1 \tag{2}$$

where n is the spatial dimension of the domain A. The following quantities will be defined:

$$\pi(\mathbf{u}) = U(\mathbf{u}) - \int_{A_\sigma} \overline{T}_i u_i \,\mathrm{d}s \tag{3}$$

$$U(\mathbf{u}) = \frac{1}{2}\int_A C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\,\mathrm{d}A \tag{4}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{5}$$

$$\mathbf{u} = \sum_m \mathbf{q}_m^T \mathbf{f}_m(\mathbf{x}) \tag{6}$$

$$\hat{\mathbf{u}}_0 = \sum_m \hat{\mathbf{q}}_m^T \mathbf{f}_m(\mathbf{x}) \tag{7}$$

$$\mathbf{u}^* = \sum_m \mathbf{q}_m^* \mathbf{f}_m(\mathbf{x}) \tag{8}$$

where π is the potential energy; U, the strain energy; C_{ijkl}, the elastic coefficients (positive definite), ε_{ij}, the strain components and $\mathbf{f}_m$, the interpolation functions which have non-vanishing value only over a finite number of adjacent elements and are chosen that $\mathbf{u}$ is at least a continuous function for arbitrary $\mathbf{q}_m$, the nodal generalized coordinates at the mth nodes. As an example, in a plane elasticity problem using triangular elements, each with three nodes, $\mathbf{f}_m$ can be simply the pyramid function around nodal m, i.e. $\mathbf{f}_m$ is unity at the mth node, is a linear function within each element adjacent to the mth node and is zero elsewhere. The vector $\mathbf{q}_m$ is the generalized coordinates at the mth node and $\hat{\mathbf{q}}_m$ is the corresponding generalized coordinates of the exact solution $\mathbf{u}_0$. For example, if $\mathbf{q}_m$ represents the nodal value of $\mathbf{u}$, then $\hat{\mathbf{q}}_m = \mathbf{u}_0$ at the mth node. The vector $\mathbf{q}_m^*$ represents the generalized coordinates of the finite element solution obtained by minimization of π with respect to all $\mathbf{q}_m$ subjecting to the constraint that $\mathbf{q}_m = \hat{\mathbf{q}}_m$ over the portion of the boundary where $\mathbf{u}$ is prescribed.

It has been shown [1] that

$$U(\mathbf{u}^* - \mathbf{u}_0) \le U(\hat{\mathbf{u}}_0 - \mathbf{u}_0) \tag{9}$$

and

$$\left[\int_A (\mathbf{u}^* - \mathbf{u}_0)^2 \,\mathrm{d}A\right]^{\frac{1}{2}} \le c_0[U(\mathbf{u}^* - \mathbf{u}_0)]^{\frac{1}{2}} \tag{10}$$

where c_0 is twice of the lowest non-zero vibration frequency hence is a positive constant. It is clear that the rate of convergence of the finite element solution is bounded by the rate which $[U(\hat{\mathbf{u}}_0 - \mathbf{u}_0)]^{\frac{1}{2}}$ is approaching zero. If the interpolation functions are so chosen such that equation (6) can exactly represent any polynomial of pth degree, it can be shown that

$$\begin{aligned} |\varepsilon_{ij}(\hat{\mathbf{u}}_0 - \mathbf{u}_0)| &\le c_1 r^{\alpha-1} && \text{in } A_1 \\ |\varepsilon_{ij}(\hat{\mathbf{u}}_0 - \mathbf{u}_0)| &\le c_1 \frac{h^p}{r^{p+1-\alpha}} && \text{in } A - A_1 \end{aligned} \tag{11}$$

where A_1 is the domain covered by the elements adjacent to point R, c_1 is some positive constants and h is the maximum size of the elements. Using the expression in equations (4) and (9), we have

$$
\begin{aligned}
U(\mathbf{u}^* - \mathbf{u}_0) &\le c_1^2 c_2 \left[\int_{A_1} r^{2\alpha-2}\, \mathrm{d}A + h^{2p} \int_{A-A_1} \frac{\mathrm{d}A}{r^{2(p+1-\alpha)}} \right] \\
&\le c_1^2 c_2 \left[c_3 r_{\max}^{2\alpha-2+n} + c_4 h^{2p} \frac{1}{r_{\min}^{2(p+1-\alpha)-n}} \right]
\end{aligned} \tag{12}
$$

where c_2 is a positive constant depending on the magnitude of C_{ijkl} only, c_3 and c_4 are positive constants depending on the geometry and the arrangement of the finite element mesh, $r_{\max}$ and $r_{\min}$ are, respectively, the maximum and the minimum radial distance from R to the boundaries of A_1. A substitution of equation (12) into equation (10) yields

$$
\left[\int_A (\mathbf{u}^* - \mathbf{u}_0)^2\, \mathrm{d}A \right]^{\frac{1}{2}} \le c_0 c_1 \sqrt{c_2} \left[c_3 r_{\max}^{2\alpha-2+n} + \frac{c_4 h^{2p}}{r_{\max}^{2(p+1-\alpha)-n}} \right]^{\frac{1}{2}}. \tag{13}
$$

The contribution of the square of the error from the elements immediately adjacent to the point of singularity is of order $r_{\max}^{2\alpha-2+n}$ and the contribution from the rest of the domain is of order $h^{2p}/r_{\min}^{2(p+1-\alpha)-n}$. From equation (12), it is clear that the main contributor for the latter part is from the elements close to the singularity. In practice, $r_{\max}$, $r_{\min}$ and h are of the same order of magnitude, i.e.

$$
r_{\max} \sim r_{\min} \sim h \tag{14}
$$

we can write equations (12) and (13) as

$$
U(\mathbf{u}^* - \mathbf{u}_0) \le c h^{2\alpha-2+n} \tag{15}
$$

$$
\left[\int_A (\mathbf{u}^* - \mathbf{u}_0)^2\, \mathrm{d}A \right]^{\frac{1}{2}} \le \sqrt{(c)}\, h^{\alpha-1+(n/2)} \tag{16}
$$

where

$$
c = c_1^2 c_5 \tag{17}
$$

and c_5 is a finite positive constant which depends on the value of the elastic constants, the geometry and the arrangement of the mesh. It is realized that the constant c_1 which is defined in equation (11) depends on the behavior of $\mathbf{u}$ near the point of singularity. From equations (15) and (16), it is clear that the order of convergence of $\mathbf{u}$ is controlled by the order of the singularity, rather than by the order of the polynomial used for the interpolation provided that equation (2) holds.

Consider the problem of plane elasticity with a sharp crack at the crack tip the displacement distribution is

$$
\mathbf{u}_0 = \begin{bmatrix} u \\ v \end{bmatrix} = \frac{k_{\mathrm{I}}(2r)^{\frac{1}{2}}}{8G} \begin{bmatrix} (2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \\ (2\kappa+1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2} \end{bmatrix} + \frac{k_{\mathrm{II}}(2r)^{\frac{1}{2}}}{8G} \begin{bmatrix} (2\kappa+3)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \\ -(2\kappa-3)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \end{bmatrix} + O(r) \tag{18}
$$

where k_{I}, k_{II} are the stress intensity factors and κ takes the value $3-4\nu$ for plane strain state and $(3-\nu)/(1+\nu)$ for plane stress state. In this case, we have

$$
\begin{aligned}
n &= 2 \\
\alpha &= \tfrac{1}{2}
\end{aligned} \tag{19}
$$

for any $p \geq 1$, equation (2) is satisfied. According to equations (15) and (16), we have

$$U(\mathbf{u}^* - \mathbf{u}_0) \leq ch \tag{20}$$

$$\left[\int_A (\mathbf{u}^* - \mathbf{u}_0)^2 \, dA \right]^{\frac{1}{2}} \leq \sqrt{ch} \tag{21}$$

i.e. the convergence in strain energy is only of order h and the convergence in displacements is of order $h^{\frac{1}{2}}$. A finite plate with edge cracks shown in Fig. 1(a) has been investigated numerically. It should be noted that

$$U(\mathbf{u}^* - \mathbf{u}_0) = U(\mathbf{u}_0) - U(\mathbf{u}^*) \tag{22}$$

in this case.

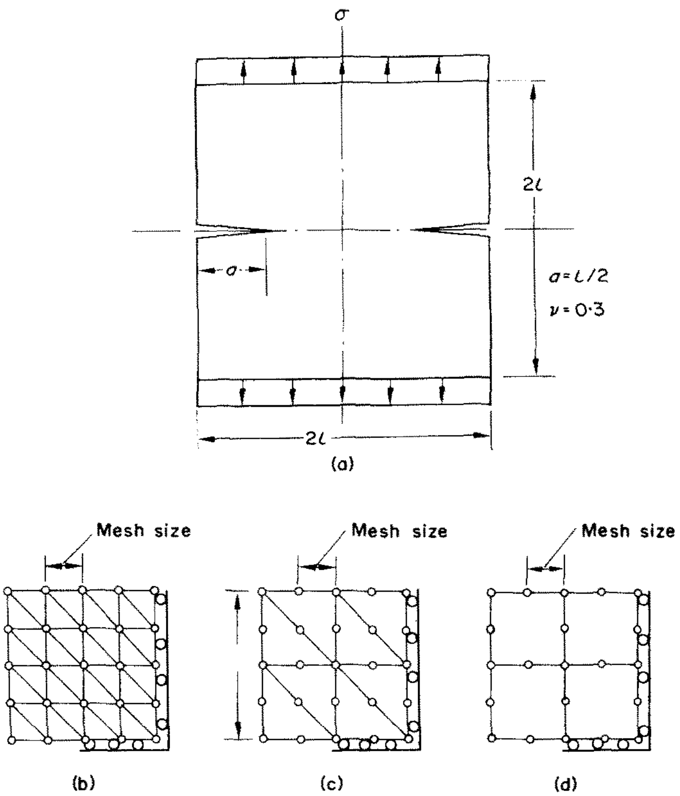

FIG. 1. Geometry and element subdivision.

The convergence characteristics of the finite element solution of the crack problem by the assumed stress hybrid model [8] is similar to that of the conventional compatible displacement model. A proof of such characteristics can also be established by using a procedure similar to that of Ref. 5, except that in establishing the error in strain energy one has to account for the singularity similar to equation (11). The details of such proof shall not be given here.

To verify the above development numerical solutions obtained by two finite element models are used: (1) compatible displacement model using triangular elements [Figs. 1(b) and (c)]. In the case of Fig. 1(b), linear interpolation functions are used for the displacements. If there were no crack, the convergence rate in strain energy would be of order h^2. In the case of Fig. 1(c), complete quadratic interpolation functions ($p = 2$) are used for the displacements. If there were no crack, the convergence rate in strain energy would be of order h^4, (2) assumed stress hybrid model using rectangular elements [Fig. 1(d)]. The quadratic displacements are used along the interelement boundaries and both cases of complete quadratic and complete cubic equilibrating stresses are used within the elements. If there were no crack, the convergence rate in strain energy in both cases will also be at least of order h^4.

The numerical results are given in Fig. 2. The convergence rate in strain energy for all the cases are indeed linear function of h as predicted in equation (20).

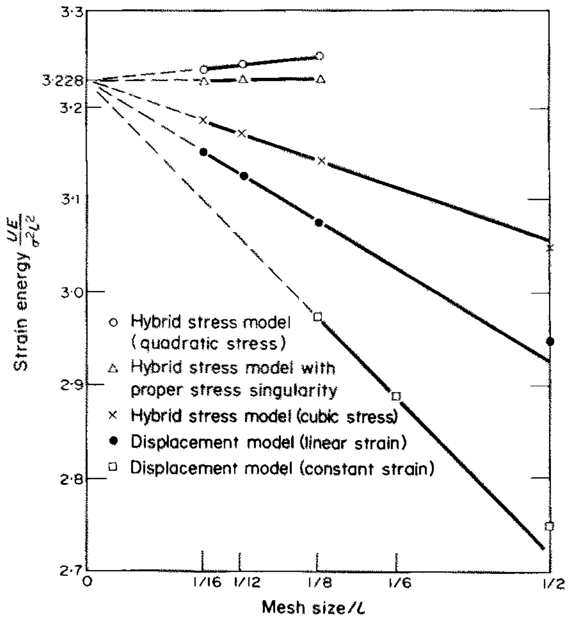

FIG. 2. Convergence in strain energy.

CONVERGENCE OF STRESS INTENSITY FACTOR k

The stress intensity factor k is related to the strain energy release rate φ, i.e. the rate of change in the strain energy due to crack extension. For example, for the mode I type crack in the plane stress and plane strain problems

$$\varphi_I = \frac{dU}{da} = \frac{(\kappa+1)\pi}{8G} k_I^2. \tag{23}$$

It has been concluded by several investigators [9, 10] that the most accurate finite element scheme for determining the stress intensity factor is by means of the evaluation of the energy release rate. We shall examine in the following the rate of convergence in stress intensity factor.

Let us examine first the constant c, i.e. the slope of convergence in strain energy in the crack analysis by the conventional finite element displacement method. From equations (11) and (18), it is clear that c_1 is linearly proportional to the stress intensity factor, k_{I} or k_{II}. For simplicity, it will be denoted by k. Since for an edge crack in an infinite plate k is proportional to the square root of the crack length, a, from equation (17) we can write

$$c \cong c_6 a \tag{24}$$

where c_6 is a constant. This linear relation is verified by a numerical solution using the displacement model as shown in Fig. 3.

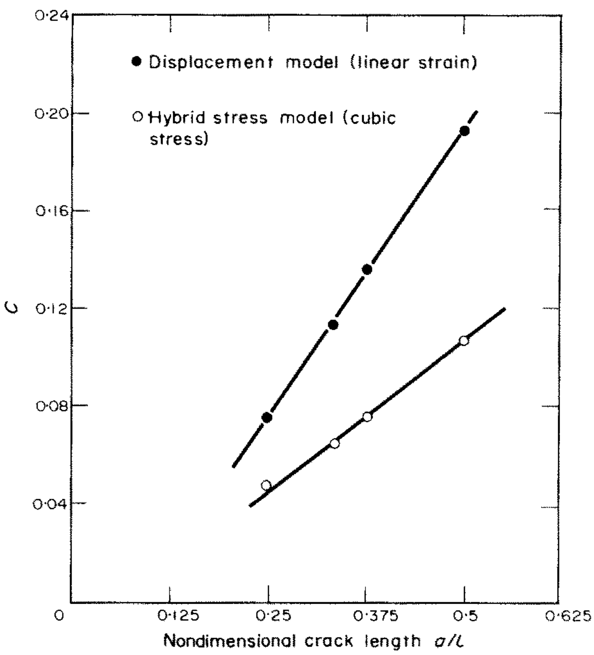

FIG. 3. Slope of convergence in strain energy vs. crack length.

In practice, let

$$\begin{aligned} U^*(a) &= U(\mathbf{u}^*) \\ U_0(a) &= U(\mathbf{u}_0) \end{aligned} \tag{25}$$

for a given crack length. Then an approximate stress intensity factor is given by

$$k^* = \left[\frac{U^*(a+\Delta a) - U^*(a)}{[(\kappa+1)\pi/8G]\Delta a}\right]^{\frac{1}{2}}. \tag{26}$$

Using equations (22) and (25), we can write equation (20) in the form

$$U^*(a) = U_0(a) + c(a)h + O(h^2). \tag{27}$$

From equation (26)

$$k^* = \left[\frac{U_0(a+\Delta(a) - U_0(a)}{[(\kappa+1)\pi/8G]\Delta a} + \frac{c(a+\Delta a)-c(a)}{[(\kappa+1)\pi/8G]\Delta a}h + O(h^2)\right]^{\frac{1}{2}}. \tag{28}$$

For Δa sufficiently small, we have

$$\begin{aligned} k^* &= \left[k^2 + \frac{8c_6 G}{(\kappa+1)\pi}h + O(h^2)\right]^{\frac{1}{2}} \\ &= k + \frac{1}{2}\frac{8Gc_6}{k(\kappa+1)\pi}h + O(h^2) \end{aligned} \tag{29}$$

i.e. the rate of convergence in the stress intensity factor determined by energy release rate is also of order h.

REMARKS

It should also be noted that equations (15) and (16) are also valid for the problems of through crack in a Kirchhoff-type plate under bending, provided that α in equation (1) is replaced by $\alpha+1$, and the interpolation f_m are chosen such that not only the displacement function itself, but its first partial derivatives are also continuous over the entire domain.

It has been shown in equations (15) and (16) that the rate of convergence is independent of p, the order of the complete polynomial used for the interpolation functions. That is, under the restriction of equation (2), the use of higher accuracy element (i.e. the use of higher order of polynomial for interpolation function) cannot improve the rate of convergence of the finite element solution.

From equation (13), it is clear that the error from the elements immediately adjacent to the point of singularity is of the same order as that of the rest of the elements. This is because large error also comes from the row of elements next to those immediately adjacent to the point of singularity. In order to improve the rate of convergence, one has to include in the interpolation functions terms which can account for the proper singularity so that a smaller error in the approximation of the strains (and the stresses) than that given in equation (11) can be achieved. These special interpolation functions should be used not only for the elements immediately adjacent to the singular point but also for those in a finite region around it. For example when the special elements are confined only to the immediate neighborhood of the crack tip then in the limiting case when the size of these elements approaches zero the solution is reduced back to that of the conventional method for which the singular terms are not included. This point can perhaps explain the fact that in the various attempts [11, 12] in solving the crack tip stress distribution problem when the size of the special elements are only 1 or 2 per cent of the crack length there still remain a few per cent errors in the finite element solution of the stress intensity factors. In the work by Hilton and Hutchinson [11] the special element is a single circle of radius equal to the 2 per cent of crack length. This element contains only the correct singular function in both r and θ. However, in the remaining elements only constant strain triangular elements are used. In the work by Tracey [12] a ring of triangular elements which contains $1/\sqrt{r}$ stress singularity are used at the crack tip. Again for the remaining elements only bilinear interpolation functions are used.

It is the opinion of the present authors that a more suitable approach in taking the singularity into account is to include the special singular function in a finite region which contains the singular functions correct in both r and θ variations. Such region must not be too small in comparison to the crack length.

For example, in Fig. 4 when a region bounded by $x = \pm x_1$ and $y = \pm y_1$, is defined where x_1 and y_1 are some appropriate lengths. The displacements in this region are then expressed as

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_s(x^2 - x_1^2)(y^2 - y_1^2) \tag{30}$$

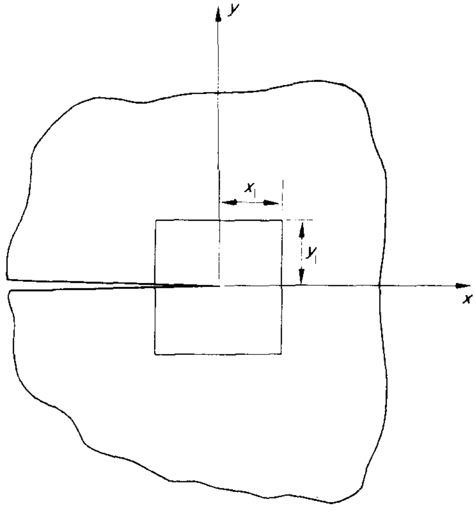

FIG. 4. Schematic diagram of the region for the special elements.

where $\mathbf{u}_s$ is in the form of the first term of equation (18) and $\mathbf{u}_r$ are nonsingular terms. The inclusion of $\mathbf{u}_r$ may be achieved either by the use of higher order interpolation functions or by further subdividing the region into smaller elements within each of which a simple interpolation functions are used. The multiplication factor for $\mathbf{u}_s$ in equation (30) is chosen to insure the compatibility of $\mathbf{u}$ at the boundary of the special region.

In Ref. [13] another method has been developed for evaluating the elastic stress intensity factors based on a hybrid stress model. In such scheme special eight-node rectangular elements are formulated which include special stress terms representing the correct singularity behavior at the crack tip in addition to some non-singular stress terms. It has been demonstrated in one example that the error in the calculated stress intensity factor by this method is only 0·3 per cent when the special region is within a radius of about $l/4$ of the crack length around the crack tip. The strain energy for the cracked plate of Fig. 1 has also been evaluated by this hybrid stress model and plotted for comparison in Fig. 2. It is seen that the resulting strain energy already coincides with the exact value when the element size used is only $l/4$. It should be noted that the mesh size defined in Fig. 1 is equal to half of the element size for this case.

The solution by Yamamoto and Tokuda [14] involves a superposition of the classical solution and the finite element solution hence is, in a sense, a scheme which extends the special singular function to the entire region. He was able to obtain stress intensity factors with less than 0·1 per cent error.

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Абстракт—Для задач с сингулярностями, очень часто приводится контроль скорости сходимости метода конечного элемента, путем исследования характера решения вблизи точек сингулярности. Для не точно обращенных сингулярностей, регулярный элемент, о так называемой большой точности, не может повышать скорости сходимости.